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LETTER TO THE EDITOR

Bose realization of a non-canonical Heisenberg algebra

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Abstract. We find out the Bose realization of a generalized Heisenberg algebra, in which the bracket of the annihilation and creation operators is proportional to a polynomial function of the number operator. The eigenvalues of the corresponding oscillator are derived in a special case. We stress also the connection between non-canonical commutation relations and q-algebras.

It is well known that the ususal (canonical) commutation relations between position and momentum operators, $\hat{x}$ and $\hat{p}$, were developed by Heisenberg. Less well known is that Heisenberg himself proposed, three decades ago, to generalize the commutation rules to a non-canonical form [1]. This new idea by Heisenberg was subsequently developed by some authors [2–5] and applied to various physical problems, such as mass variation with respect to coordinates [3], coherent states for a generic potential [4] and high-energy physics [5].

A specific form of a non-canonical commutation relation for $\hat{x}$ and $\hat{p}$ is given by [5]

$$[\hat{x}, \hat{p}] = \mathrm{i} f(\hat{H}) \tag{1}$$

where $f(\hat{H})$ is an arbitrary Hermitian function of the Hamiltonian $\hat{H}$. If the system under consideration contains some small parameter α‖, the operator function $f(\hat{H})$ can be approximated (to the first order in α) by

$$f(\hat{H}) = \hbar + \alpha\hat{H} \tag{2}$$

so that (1) becomes

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar + \mathrm{i}\alpha\hat{H}(\hat{x}, \hat{p}). \tag{3}$$

Another example of a non-canonical commutation relation in Fock space is provided by the Q-algebra by Kuryshkin [6]

$$\hat{A}\hat{A}^{+} - Q\hat{A}^{+}\hat{A} \equiv [\hat{A}, \hat{A}^{+}]_Q = \hat{I} \tag{4}$$

$(Q \in [-1, +\infty), Q \neq 0)$.

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‖ For instance, in [5] it is $\alpha = l/c$, where l is an elementary length and c the speed of light.

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As is well known, a great deal of interest about generalized brackets of the kind (4) has arisen recently in connection with quantum groups (for a review and exhaustive bibliography, see, for example, [7]), which are essentially based on the q-deformed Heisenberg-Weyl algebra

$$[\hat{A}, \hat{A}^{+}]_q \equiv \hat{A}\hat{A}^{+} - q\hat{A}^{+}\hat{A} = \hat{q}^{\hat{n}} \tag{5}$$

with q a parameter.

We want to stress that, although, *a priori*, there seems to be no connection between non-canonical commutation rules of the type (1) and the generalized brackets (4) and (5), actually they are deeply interrelated. Let us indeed prove that, for the harmonic oscillator, relation (3) in Hilbert space leads to the Q-algebra (4) for Fock operators (for a preliminary discussion at this point, see [8]).

For a harmonic oscillator Hamiltonian $\hat{H}$ the non-canonical commutations rule (3) becomes

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar + \frac{\mathrm{i}\alpha}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2). \tag{6}$$

The Fock representation of $\hat{x}$ nd $\hat{p}$ in terms of $\hat{A}$, $\hat{A}^{+}$ reads, in this case

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega\Lambda}}(\hat{A} + \hat{A}^{+}) \tag{7a}$$

$$\hat{p} = -\mathrm{i}\sqrt{\frac{\hbar m\omega}{2\Lambda}}(\hat{A} - \hat{A}^{+}) \tag{7b}$$

where Λ is a suitable scale factor needed to get a closed algebra for $\hat{A}$, $\hat{A}^{+}$.

Replacing (7) in (6) we easily find

$$\hat{A}\hat{A}^{+}\left(1 - \frac{\alpha\hbar\omega}{2}\right) - \left(1 + \frac{\alpha\hbar\omega}{2}\right)\hat{A}^{+}\hat{A} = \Lambda \tag{8}$$

whence the Q-algebra (4) immediately follows, with

$$Q = \frac{1 + \alpha\hbar\omega/2}{1 - \alpha\hbar\omega/2} \tag{9}$$

provided that

$$\Lambda = 1 - \frac{\alpha\hbar\omega}{2}. \tag{10}$$

Due to the connection established above, it is worth investigating non-canonical Heisenberg algebras both in the ususal Hilbert space and in Fock space. For instance, a possible generalization of the q-deformed algebra (5) is provided by [9]

$$\hat{A}\hat{A}^{+} - Q\hat{A}^{+}\hat{A} = f(\hat{n}) \tag{11}$$

where $\hat{n}$ is the ususal number operator and the function $f(\hat{n})$ is a suitable (smooth) function†.

Of course, it is not a simple task, in general, to deal with (11), even for $f(\hat{n})$ a regular function. We want to show here how to approach this problem in the case $Q = 1$ and $f(\hat{n})$ a polynomial of degree k in $\hat{n}$, i.e.

$$[\hat{A}, \hat{A}^{+}] = P(\hat{n}) = 1 + \alpha_1\hat{n} + \alpha_2\hat{n}^2 + \ldots + \alpha_k\hat{n}^k \tag{12}$$

with α_i ($i = 1, \ldots, k$) real numbers.

† If $f(\hat{n}) = q^{\hat{n}}$, we get the two-parameter quantum algebra considered in [10].

Let us find a boson realization of the annihilation and creation operators $\hat{A}$, $\hat{A}^+$ obeying the commutation rule (12).

We apply the bosonization method [8, 9] and seek $\hat{A}$, $\hat{A}^+$ in the form

$$\hat{A} = F(\hat{n}+1)\hat{a} \qquad \hat{A}^+ = \hat{a}^+ F(\hat{n}+1) \tag{13}$$

where $\hat{a}$, $\hat{a}^+$, $\hat{n} = \hat{a}^+\hat{a}$ are boson operators, satisfying the ususal commutation relations.

Then, from (12) we get the following equation for the operator function $F(\hat{n})$:

$$(\hat{n}+1)F^2(\hat{n}+1) - \hat{n}F^2(\hat{n}) = P(\hat{n}). \tag{14}$$

By putting $\hat{n}F^2(\hat{n}) = L_n$, we have

$$L_{n+1} - L_n = P(\hat{n}) \tag{15}$$

i.e. a set of difference equations with initial condition $L_0 = 0$.

The solution of (15) has the form

$$L_{n+1} = (\hat{n}+1) + \alpha_1 \frac{\hat{n}(\hat{n}+1)}{2} + \alpha_2 \frac{\hat{n}(\hat{n}+1)(2\hat{n}+1)}{6} + \alpha_3 \left[\frac{\hat{n}(\hat{n}+1)}{2}\right]^2 + \alpha_4 \frac{\hat{n}(\hat{n}+1)(2\hat{n}+1)(3\hat{n}^2+3\hat{n}-1)}{30} + \ldots + \alpha_k S_k \tag{16}$$

where

$$S_k = \frac{\hat{n}^{k+1}}{k+1} + \frac{\hat{n}^k}{2} + \frac{1}{2}\binom{k}{1}\beta_1 \hat{n}^{k-1} - \frac{1}{4}\binom{k}{3}\beta_2 \hat{n}^{k-3} + \ldots \tag{17}$$

with the β_i's being Bernouilli numbers. From (17), (13) we get the explicit expressions of the operators $\hat{A}$, $\hat{A}^+$:

$$\hat{A} = \left[1 + \alpha_1 \frac{\hat{n}}{2} + \alpha_2 \frac{\hat{n}(2\hat{n}+1)}{6} + \alpha_3 \frac{\hat{n}^2(\hat{n}+1)}{4} + \alpha_4 \frac{\hat{n}(2\hat{n}+1)(3\hat{n}^2+3\hat{n}-1)}{30} + \ldots\right]^{1/2} \hat{a} \tag{18a}$$

$$\hat{A}^+ = \hat{a}^+ \left[1 + \alpha_1 \frac{\hat{n}}{2} + \alpha_2 \frac{\hat{n}(2\hat{n}+1)}{6} + \alpha_3 \frac{\hat{n}^2(\hat{n}+1)}{4} + \alpha_4 \frac{\hat{n}(2\hat{n}+1)(3\hat{n}^2+3\hat{n}-1)}{30} + \ldots\right]^{1/2}. \tag{18b}$$

The operators $\hat{A}$, $\hat{A}^+$ are true (non-canonical) annihilation–creation operators, since their action on the Fock states is given by

$$\hat{A}|0\rangle = |0\rangle \qquad \hat{A}^+|0\rangle = |1\rangle \tag{19}$$

$$\hat{A}|n\rangle = \sqrt{n}\left[1 + \alpha_1 \frac{n-1}{2} + \alpha_2 \frac{(n-1)(2n-1)}{6} + \alpha_3 \frac{n^2(n-1)^2}{4} + \ldots\right]^{1/2} |n-1\rangle \tag{20a}$$

$$\hat{A}^+|n\rangle = \sqrt{n+1}\left[1 + \alpha_1 \frac{n}{2} + \alpha_2 \frac{n(2n+1)}{6} + \alpha_3 \frac{n^2(n+1)^2}{4} + \ldots\right]^{1/2} |n+1\rangle. \tag{20b}$$

The above relations provide us straightforwardly with the commutation and anti-commutation rules for $\hat{A}$, $\hat{A}^+$. We have

$$\hat{A}\hat{A}^+=(\hat{n}+1)\bigg[1+\alpha_1\frac{\hat{n}}{2}+\alpha_2\frac{\hat{n}(2\hat{n}+1)}{6}+\alpha_3\frac{\hat{n}^2(\hat{n}+1)}{4}$$
$$+\alpha_4\frac{\hat{n}(2\hat{n}+1)(3\hat{n}^2+3\hat{n}-1)}{30}+\ldots\bigg] \tag{21a}$$

$$\hat{A}^+\hat{A}=\hat{n}\bigg[1+\alpha_1\frac{\hat{n}-1}{2}+\alpha_2\frac{(\hat{n}-1)(2\hat{n}-1)}{6}+\alpha_3\frac{(\hat{n}-1)^2\hat{n}}{4}$$
$$+\alpha_4\frac{(\hat{n}-1)(2\hat{n}-1)(3\hat{n}^2-3\hat{n}+2)}{30}+\ldots\bigg] \tag{21b}$$

whence one easily recovers (12) and gets

$$\{\hat{A},\hat{A}^+\}=(2\hat{n}+1)+\alpha_1\hat{n}^2+\alpha_2\frac{\hat{n}(2\hat{n}+1)}{3}+\alpha_3\frac{\hat{n}^2(\hat{n}^2+1)}{2}$$
$$+\alpha_4\hat{n}\frac{12\hat{n}^4+26\hat{n}^2-9\hat{n}+1}{30}. \tag{22}$$

We want now to construct the Fock representation according to the relations

$$\hat{A}=\sqrt{\frac{1}{\lambda\hbar}}\left(\sqrt{m\omega}\,\hat{x}+\mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right)$$
$$\hat{A}^+=\sqrt{\frac{1}{\lambda\hbar}}\left(\sqrt{m\omega}\,\hat{x}-\mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right) \tag{23}$$

where λ is a scale factor to be determined.

The commutation relationof $\hat{x}$, $\hat{p}$ is easily found and reads

$$[\hat{x},\hat{p}]=\frac{\mathrm{i}}{2}\lambda\hbar P(\hat{n}). \tag{24}$$

Moreover, from (22) we get for the Hamiltonian $\hat{H}$

$$\hat{H}=\frac{\lambda\hbar\omega}{4}\{\hat{A},\hat{A}^+\}=\frac{\lambda\hbar\omega}{4}\left[2\hat{n}+1+\alpha_1\hat{n}^2+\alpha_2\frac{\hat{n}(2\hat{n}+1)}{3}+\alpha_3\frac{\hat{n}^2(\hat{n}^2+1)}{2}+\ldots\right]. \tag{25}$$

The right-hand side of the above equation is a polynomial of degree $(k+1)$ in $\hat{n}$ and therefore it is possible, in principle, to find a solution $\hat{n}(\hat{H})$.

This is required in order to put the commutator (24) in the form (1) of a non-canonical Heisenberg algebra. A way to solve (25) is to express $\hat{n}(\hat{H})$ as an expansion in powers of $\hat{H}$, namely

$$\hat{n}(\hat{H})=f_0+f_1\hat{H}+f_2\hat{H}^2+\ldots+f_l\hat{H}^l+\ldots. \tag{26}$$

The parameter f_0 satisfies the equation

$$2f_0+1+\alpha_1f_0^2+\alpha_2\frac{f_0(2f_0+1)}{3}+\alpha_3\frac{f_0^2(f_0^2+1)}{2}+\ldots=0 \tag{27}$$

whereas the other f_l's ($l=1,2,\ldots$) obey equations of the type

$$\frac{4}{\lambda\hbar\omega}=F_1(f_1,f_0,\alpha_1,\ldots,\alpha_k)$$

$$F_2(f_2,f_1,f_0,\alpha_1,\ldots,\alpha_k)=0 \tag{28}$$

$$\ldots\ldots$$

Replacing (26) in (24) we get therefore

$$[\hat{x},\hat{p}]=\frac{\mathrm{i}}{2}\lambda\hbar[(1+\alpha_1f_0+\alpha_2f_0^2+\ldots+\alpha_kf_0^k)+\mu_1\hat{H}+\mu_2\hat{H}^2+\ldots.+\mu_k\hat{H}^k+\ldots] \tag{29}$$

where μ_j are functions of $f_1,f_2,\ldots,\alpha_1,\ldots,\alpha_k$, $4/\lambda\hbar\omega$. Then, it is easy to see that (29) takes exactly the form (1) if

$$\frac{\lambda}{2}(1+\alpha_1f_0+\alpha_2f_0^2+\ldots+\alpha_kf_0^k)=1 \tag{30}$$

with

$$\varphi(\hat{H})=\tfrac{1}{2}(\mu_1+\mu_2\hat{H}+\ldots+\mu_k\hat{H}^{k-1}+\ldots). \tag{31}$$

Condition (30) (i.e. the requirement of a non-canonical commutation relation for $\hat{x},\hat{p}$) fixes the value of the scale factor λ.

Let us work out an explicit example by considering the case $k=1$, i.e.

$$[\hat{A},\hat{A}^+]=1+\alpha\hat{n}. \tag{32}$$

From (18), (22) we get

$$\hat{A}=\left(1+\frac{\alpha\hat{n}}{2}\right)^{1/2}\hat{a}\qquad \hat{A}^+=\hat{a}^+\left(1+\frac{\alpha\hat{n}}{2}\right)^{1/2} \tag{33}$$

$$\hat{A}\hat{A}^+=(\hat{n}+1)\left(1+\frac{\alpha}{2}\hat{n}\right)\qquad \hat{A}^+\hat{A}=\hat{n}\left[1+\frac{\alpha}{2}(\hat{n}+1)\right] \tag{34}$$

$$\{\hat{A},\hat{A}^+\}=2\hat{n}+1+\alpha\hat{n}^2. \tag{35}$$

The Hamiltonian $\hat{H}$ reads

$$\hat{H}=\frac{\lambda\hbar\omega}{4}(2\hat{n}+1+\alpha\hat{n}^2) \tag{36}$$

whence

$$\hat{n}=\frac{-1+\sqrt{(1-\alpha)+(4\hat{H}\alpha/\lambda\hbar\omega)}}{\alpha}. \tag{37}$$

The commutator (24) of $\hat{x}$ and $\hat{p}$ then becomes

$$[\hat{x},\hat{p}]=\frac{\mathrm{i}}{2}\hbar\lambda(1+\alpha\hat{n})=\mathrm{i}\hbar\sqrt{\frac{\lambda^2}{4}(1-\alpha)+\frac{\lambda\alpha\hat{H}}{\hbar\omega}} \tag{38}$$

which, for $(\lambda^2/4)(1-\alpha)=1$, provides us with the general non-canonical commutation relation

$$[\hat{x},\hat{p}]=\mathrm{i}\hbar\sqrt{1+\frac{2\alpha\hat{H}}{\hbar\omega(1-\alpha)}}. \tag{39}$$

The eigenvalues of the corresponding harmonic oscillator are given by

$$E_n = \frac{\hbar\omega}{(1-\alpha)}\left(n + \frac{1}{2} + \frac{\alpha}{2} n^2\right). \tag{40}$$

It is immediately clear from the above formula that α must satisfy the constraint $0 \leqslant \alpha < 1$. For $\alpha = 0$, obviously, one recovers the standard relations of the ususal harmonic oscillator.

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